Signal-Matched Wavelet Design via Lifting using Optimization Techniques

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***Abstract*—This paper proposes design of signal-matched wavelets via lifting. The design is modular owing to lifting framework wherein both predict and update stage polynomials are obtained from the given signal. Successive predict stages are designed using the least squares criterion, while the update stages are designed with total variation minimization on the wavelet approximation coefficients. Different design strategies for compression and denoising are presented. The efficacy of matched-wavelets is illustrated on transform coding gain and signal denoising.**

***Index Terms*—Signal-matched wavelet system; lifting scheme; optimization techniques.**

I. Introduction

Design of multirate filterbanks and wavelets is an active research area explored extensively by applied mathematicians and signal processing community. Wavelets have been applied successfully in many areas applications including compression, denoising, pattern matching, watermarking, biomedical signal and image processing, texture analysis, traffic modeling, etc. Compared to the traditional Fourier-based analysis, wavelet analysis provides an option to choose different basis. Since the basis here is not unique, it is natural to seek a wavelet that is best in a particular context for a given signal.

Design of signal-adapted or signal-matched wavelets has been addressed with lifting in [1-10]. The lifting technique involves alternate predict and update steps. Although it is easy to find the prediction stage filters, finding an update filter offers a real challenge. One of the criteria used in the literature to find the update filters is the minimization of reconstruction error of even and odd indexed samples [1]. In [2,3], the update first structure with adaptation of the update step is used. The update filter is changed based on the local gradient information such that sharp variations in the signal get less smoothened than the more homogenous regions. Similar update method is used in [4]. In [5], a nonseparable lifting is used on images with regularity conditions imposed. In [6], directional interpolation is used with coefficients of interpolation filter to optimize to adapt to statistical property of image. In [7], authors have designed wavelets by minimizing the difference between BWT (Block Wavelet Transform) and KLT (Karhunen-Loève Transform) of signal. In [8], orthogonal IIR (Infinite Impulse Response) filterbank is designed using allpass filter in the lifting steps. In [9], geometry of the given image is used to design new wavelet via lifting leading to local and anisotropic filters. [10] has designed nonseparable filterbanks which are pixel-wise adapted according to local image feature.

In this paper, we propose to design signal-matched wavelets using lifting wherein both predict and update stage polynomials are obtained from a given signal. Successive predict stages are designed using the least squares criterion, while the update stages are designed with total variation minimization on the wavelet approximation coefficients. We propose two design methods. Method-1 designs signal matched filters with no constraint of linear phase property imposed on filters, while method-2 designs linear phase scaling and wavelet filters. We test our design methods on some randomly picked speech and music clips and compare results of designed wavelets with standard wavelets on transform coding gain and signal denoising. The signal-matched wavelets are designed differently for compression (illustrated via transform coding gain) and denoising.

In [11], nonseparable wavelets are designed for images using lifting using the criterion of variance minimization in the wavelet space for the predict stage. For the update stage, reconstruction error is minimized between the input signal and the output signal after dropping the wavelet subband. The work proposed in this paper is carried out independently of [11], although it is noticed to have some similarity in the design approach. This work differs from [11] in the following ways:

- In this work, we design signal-matched wavelets for 1-D signals with 2-tap update and the predict polynomials in the powers of z and z^{-1}, respectively. This leads to the design of signal-matched 5/3 and 9/7 wavelets with one and two stages of predict-update pairs, respectively. Other variations on number of filter taps or different polynomials in z or z^{-1} will not lead to these wavelets. On the other hand, [11] designs nonseparable wavelets for images without any such focus.
- We show that signal-matched wavelets designed differently in different applications lead to better designs. Here, a different approach is proposed to design matched wavelet for denoising compared to compression.
- Also, we use total variation minimization constraint in

the update stage, while [11] uses a constraint related to the nonseparable quincunx lattice of the image.

The paper is organized as follows. In section 2, we present a brief review of lifting scheme. Section 3 presents our proposed methods on signal-matched wavelet design. Simulation results are presented in section 4. Some conclusions are presented in section 5.

II. Theory of Lifting in Brief

Lifting, also known as second generation wavelets, is a technique for either factoring existing wavelet filters into a finite sequence of smaller filtering steps or constructing new customized wavelet basis [12]. A general lifting scheme consists of three steps: Split, Predict, and Update (Refer to figure 1).

Split: In the split step, given input signal is split into two disjoint sets, generally even indexed and odd indexed samples, labeled as $x_e[n]$ and $x_o[n]$, respectively. The original signal can be recovered perfectly by interlacing or combining this even and odd indexed sample stream. The corresponding filterbank structure is also called as the Lazy wavelet system [12] and the related filterbank structure is shown in Fig. 2 with analysis filters labeled as $H_0(z) = Z\{h_0[n]\}$, $H_1(z) = Z\{h_1[n]\}$ and the synthesis filters as $F_0(z) = Z\{f_0[n]\}$, $F_1(z) = Z\{f_1[n]\}$.

Predict or Dual Lifting Step: In the predict stage, one of these two disjoint sets is predicted from the other set. For example, in figure 1(a), we predict even samples from the neighboring odd samples by using the predictor $P \equiv T(z)$. Predict stage is equivalent to applying a high-pass filter on the input signal. This step modifies the analysis high-pass and synthesis lowpass filter, without changing other filters according to the following relations:

$$H_1^{new}(z) = H_1(z) - H_0(z)T(z^2). \tag{1}$$

$$F_0^{new}(z) = F_0(z) + F_1(z)T(z^2). \tag{2}$$

Update or Primal Lifting Step: This step modifies the analysis lowpass filter and provides the coarse approximation of the signal. The update step is denoted with the symbol $U \equiv S(z)$. This is also called as the primal lifting step or simply, the lifting step. Update step only modifies the analysis low pass and synthesis high-pass filter according to the following relation:

$$H_0^{new}(z) = H_0(z) + H_1(z)S(z^2). \tag{3}$$

$$F_1^{new}(z) = F_1(z) - F_0(z)S(z^2). \tag{4}$$

One of the major advantages of lifting scheme is that each stage (predict or update) is invertible. Hence, perfect reconstruction (PR) is guaranteed.

III. Proposed Wavelet Design

In this section, we propose two methods of designing matched wavelet via lifting. In the first method, we discuss wavelet design without imposing the condition of linear phase (LP) on filters. The second method designs linear phase filters.

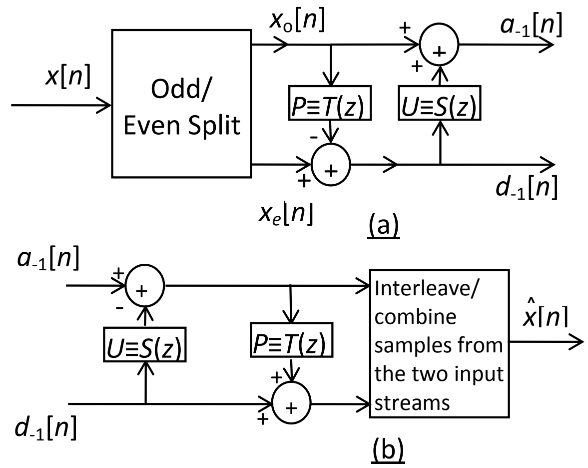


Fig. 1: Steps of Lifting: Split, Predict and Update

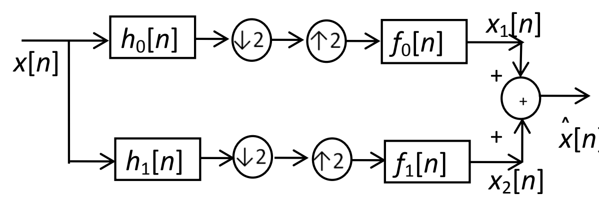


Fig. 2: Two Channel Biorthogonal Wavelet System

A. Method-1: With no constraint of LP

Let us refer to figure 3 that considers the following filters for the Lazy wavelet:

$$H_0(z) = z^{-1}, H_1(z) = z^{-2}, \tag{5}$$

$$F_0(z) = z^{-2}, F_1(z) = z^{-1}. \tag{6}$$

This set of filters gives perfect reconstruction with

$$\hat{x}[n] = x[n-3] \tag{7}$$

Starting from this, we now present our method to design predict and update stages.

1) Design of Predict Stage: The wavelet subband coefficients from the lower branch of figure 3(a) can be written as:

$$\begin{aligned} d_{-1}[n] =& x_e[n] - P_1(x_o[n]) \\ =& x[2n-2] - t_0 x[2n-1] - t_1 x[2n-3] \\ =& \sum_k h_1[k]x[2n-k] \end{aligned} \tag{8}$$

where $H_1(z) = -t_0 z^{-1} + z^{-2} - t_1 z^{-3}$. Assuming that input signal is rich in low frequency content, most of the input signal energy after decomposition should lie in the lowpass band. Hence, predict stage polynomial $T(z) = t_0 + t_1 z^{-1}$ can be obtained by minimizing the energy of the signal $d_{-1}[n]$ in the high pass band, with the following least squares criterion

$$\begin{aligned} \tilde{\mathbf{t}} &= \min_{\mathbf{t}} \|\mathbf{d}_{-1}\|_2^2 \\ &= \min_{\mathbf{t}} \|\mathbf{b} - \mathbf{A}\mathbf{t}\|_2^2 \end{aligned} \tag{9}$$

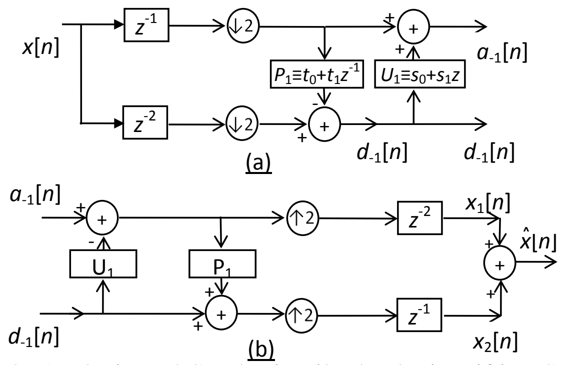


Fig. 3: Analysis and Synthesis Filterbanks in Lifting Steps

where $\mathbf{b} = \begin{pmatrix} x[2] \\ x[4] \\ x[6] \\ \vdots \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} x[3] & x[1] \\ x[5] & x[3] \\ x[7] & x[5] \\ & \vdots \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} t_0 \\ t_1 \end{pmatrix}$

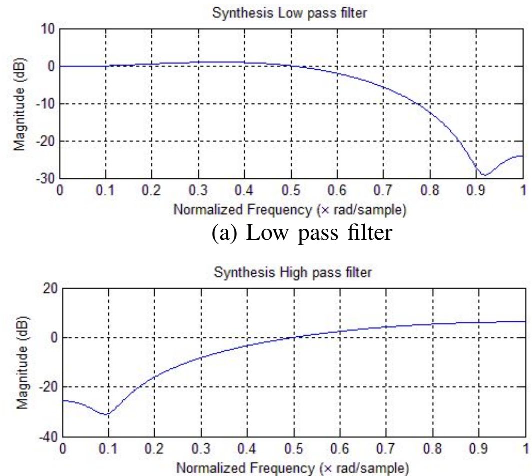
The solution of equation (9) provides the estimated polynomial $\tilde{T}(z)$, which can be used in equation (1) and (2) to update $h_1[n]$ and $f_0[n]$, respectively.

B. Design of Update stage

Next, we propose to design the update stage. We rely on the argument that the signal reconstructed from the lowpass filter branch, depicted as $x_1[n]$ in figure 2, should be the closest approximation of the input signal $x[n]$. In addition, wavelet approximation coefficients should form a smooth signal with dominantly low frequency information. Thus, the total variation of the subband signal $a_{-1}[n]$ should be minimum. This helps us in formulating the optimization criterion to determine the update stage $U_1 \equiv S_1(z) = (s_0 + s_1 z)$ as below:

$$\tilde{\mathbf{s}} = \min_{\mathbf{s}} (\|\mathbf{x} - \mathbf{x}_1\|_2^2 + \lambda \|D\mathbf{a}_{-1}\|_1) \qquad (10)$$

where $\mathbf{s} = [s_0 \ s_1]^T$, D denotes the first differencing operation, and small bold case letters denote the vector form of the corresponding time indexed signals. In order to solve (10), we note that we updated $f_0[n]$ after the previous predict stage. It should be noted that both $\mathbf{a}_{-1}$ (vector form of $a_{-1}[n]$ in Fig. 3(a)) and $\mathbf{x}_1$ (vector form of $x_1[n]$ in Fig. 3(b)) can be written explicitly in terms of $\mathbf{s}$. Equation (10) can be solved using any optimization toolbox. We used CVX, a package for specifying and solving convex programs [16,17]. Next, we update $h_0[n]$ and $f_1[n]$ using (3) and (4), respectively. This is to note that we look for global solution for the update stage compared to many of the existing methods that design the update stage using the local information. Also, the update and the predict polynomials are 2-tap and in the powers of z and z^{-1}, respectively. This leads to the design of signal-matched 5/3 and 9/7 wavelets with one and two stages of predict-update pairs, respectively. Subsequent predict and update stages can be designed iteratively using the above procedure.



(a) Low pass filter

(b) High pass filter

Fig. 4: Frequency response of synthesis end filters

C. Method-2: With LP Filter Design

A linear phase filter is symmetric or anti-symmetric about the center weight. On expanding filters $h_0[n]$ and $h_1[n]$ in terms of polynomials $T(z)$ and $S(z)$ of predict and update stage, it is noted that choosing $T(z)$ and $S(z)$ as below help with the design of linear phase filters.

$$T(z) = t_0(1 + z^{-1}) \text{ and } S(z) = s_0(1 + z) \qquad (11)$$

IV. Experiments and Result

In order to validate our results on signal-matched wavelets, we apply the proposed methods on speech and music signals. Music signals are picked randomly from [18]. One stage and two stages of predict and update are computed with and without linear phase conditions. The resulting wavelet system corresponds to the synthesis filters of lengths 5/3 (highpass/lowpass) and 9/7 (highpass/lowpass) with one and two stages, respectively. Analysis side filters for one speech and one music signal are presented in Table-1. Synthesis side filters can be obtained as per equation (2) and (4). Since the resulting wavelet system are signal-matched biorthogonal 5/3 and 9/7, it is appropriate to compare results with the standard biorthogonal 5/3 and 9/7 wavelets. In both the experiments, the value of λ in equation (10) is emipirically found to be 0.01. Frequency response of synthesis low pass and high-pass filter is shown in Fig. 4 for one of the speech signal.

Next, we apply our design method on two applications- compression via computation of transform coding gain and denoising.

A. Transform Coding Gain

Transform coding gain is a common measure used to ascertain the efficiency of the signal transform. It is defined as the ratio of error power obtained by directly quantizing input signal coefficients $x[n]$ to the error power obtained by quantizing the subband coefficients using an optimal bit allocation strategy at a given average bit rate [13]. We have used 1-level wavelet decomposition for transform coding gain.

Table-1: Matched wavelet filters of two signals

S.No.	Input Signal	Filter Coefficients
1.	Speech-1 Sampling frequency: f_s=11.025 KHz Number of samples = 2712	9/7 Filters
		$\mathbf{h}_0$=[0 -0.0004 0.0007 -0.1225 0.2578 0.7108 0.3126 -0.1593 0.0007 -0.0003]
		$\mathbf{h}_1$=[0 0.0089 -0.0183 -0.5360 1.0016 -0.4642 0.0167 -0.0086 0 0]
		5/3 Filters
		$\mathbf{h}_0$=[0 -0.1296 0.2380 0.7336 0.2648 -0.1360]
		$\mathbf{h}_1$=[0 0 0 -0.5445 1.0000 -0.5136]
2.	Music-1 Sampling frequency: f_s=11.025 KHz Number of samples = 10000	9/7 Filters
		$\mathbf{h}_0$=[0 -0.0004 0.0007 -0.1192 0.2414 0.7091 0.3293 -0.1615 0.0008 -0.0004]
		$\mathbf{h}_1$=[0 0.0090 -0.0178 -0.5569 0.9997 -0.4432 0.0181 -0.0090 0 0]
		5/3 Filters
		$\mathbf{h}_0$=[0 -0.1404 0.2791 0.7183 0.2841 -0.1413]
		$\mathbf{h}_1$=[0 0 0 -0.5029 1.0000 -0.4973]

Table-2: Results of Transform Coding Gain (in dB)

Signal	Transform Coding Gain					
	MW 9/7 LP	MW 9/7	Standard 9/7 LP	MW5/3 LP	MW 5/3	Standard 5/3 LP
Speech-1	9.9852	9.6587	**10.4833**	10.0400	9.9838	9.2843
Speech-2	5.3676	5.3629	4.7329	5.3659	5.3542	**5.3777**
Music-1	**3.6977**	3.6756	3.3889	3.6665	3.6663	3.6952
Music-2	**4.2032**	4.1938	3.6396	4.2011	4.1954	4.1987
Music-3	**15.7042**	15.6912	15.1751	15.7041	15.7041	15.3920

Table 2 presents transform coding gain results of our designed matched wavelets (MW) with and without LP condition. Results show that designed matched wavelet performs better or comparable to the corresponding standard wavelets.

B. Proposed Matched Wavelet design for denoising Application

Discrete wavelet transform not only provides compact representation for a wide class of signal, it has been proved to be a powerful tool for signal denoising. Since our lowpass filter is designed in the update stage considering that most of the signal energy will move to the low frequency branch of the filterbank, our proposed scheme of matched wavelet is suited for signals rich in low frequencies. On the contrary, noisy signal will be rich in high frequency content. Thus, we use accumulator, which is a discrete time counterpart of an integrator, on the given noisy signal $x(n)$ as below:

$$y[n] = \sum_{k=0}^{n} x[k] \tag{12}$$

where $x[n] = 0$, when $n < 0$. This step will convert input noisy signal $x[n]$ into dominantly lowpass signal $y[n]$. Resulting dominantly lowpass signal $y[n]$ is fed as input to our algorithm and wavelet filterbank is designed matched to this signal $y[n]$ [14]. After denoising as discussed in the next paragraph, we apply first difference on the successive samples of the output signal $s[n]$ to obtain the actual denoised signal $\hat{x}[n]$ according to the following relation:

$$\hat{x}[n] = s[n] - s[n-1] \tag{13}$$

We add white Gaussian noise at 5dB SNR per sample. After designing the matched system, soft-thresholding is applied on the wavelet coefficients. We have applied 3-level wavelet decomposition for denoising. All the subband coefficients are thresholded using *Bayes Shrink* threshold strategy [15] except coarsest approximation coefficients. Table 3 shows the comparison of the denoised results of speech and music signals between matched wavelets and standard biorthogonal wavelets. Peak signal to noise ratio (PSNR) is used as the performance measure for denoising. Each experiment is performed with 30 runs and the results shown here are the average over all runs.

Table-3: Results of Denoising

Signal	PSNR in dB						
	Noisy	MW 9/7 LP	MW 9/7	Standard 9/7 LP	MW 5/3 LP	MW 5/3	Standard 5/3 LP
Speech-1	12.2799	13.1411	14.6771	14.6142	13.3723	12.9791	**14.8226**
Speech-2	11.8505	12.2408	**14.0847**	11.8370	11.8474	12.1052	12.7556
Music-1	12.3899	13.9266	**16.0590**	14.1605	13.7261	13.9833	14.6508
Music-2	11.8106	12.4449	**14.2614**	11.9371	12.3360	12.4811	12.9840
Music-3	11.8734	13.7756	**15.4210**	13.4330	13.2194	13.4132	13.8639

Discussion: From Table 3, the following observations are in order:

- Signal-matched wavelet designed without LP constraint gives better results of denoising compared to the one with the LP constraint for both the 5/3 and the 9/7 wavelet.
- Signal-matched 9/7 wavelet without LP constraint is working best on most of the signals considered. The results are better compared to the standard 9/7 and 5/3 wavelets.

The above results are obvious because with linear phase condition, we are constraining the design of matched wavelet and it may deviate from the exact matching to the signal. However, the LP matched wavelet design may be useful in other applications. This is to note that denoising results on standard 9/7 and 5/3 wavelets with the method of accumulator and first difference are observed to be inferior. Hence, for the brevity of the presentation, these have not been included in the text.

V. Conclusion

In this paper, we have proposed two methods of designing signal-matched biorthogonal wavelets via lifting using optimization techniques. The proposed method designs matched wavelet system with linear phase and without linear phase constraints. In particular, we have designed signal-matched 9/7 and 5/3 wavelets with and without linear phase constraints. We applied the proposed methods on some random speech and music signals in the context of transform coding gain and signal denoising. It is emphasized that signal-matched wavelets should be designed differently for different applications. Results of signal-matched 9/7 and 5/3 wavelets are better or comparable to the corresponding standard 9/7 and 5/3 wavelets, respectively.

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